Physics 54

Wave Optics 2

It's no exaggeration to say that the undecideds could go either way.
— Pres. George H. W. Bush

Overview

When part of a wavefront is obstructed, the progress of the wave energy into the region beyond the obstruction is determined (according to Huygens's principle) by the waves emitted from points on the *unobstructed* part of the wavefront. (The rest of the energy is either reflected or absorbed by the obstruction.) The waves from these unobstructed points spread out, sending some energy into the geometric shadow region. This energy deviates from the directions of the original rays, "bending" into the shadow. This is the phenomenon of **diffraction**. It is a general property of waves.

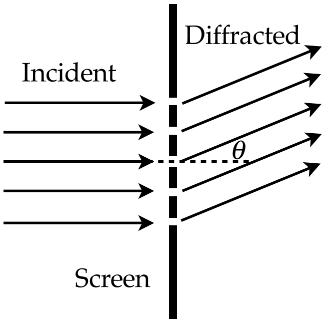
The fraction of the original unobstructed energy that diffracts into the shadow region is greater for longer wavelengths and for smaller obstacles or apertures. If the size of obstacles or apertures encountered by the wave is very large compared to the wavelength, nearly all of the energy goes in straight lines along the original rays. This is why we see fairly sharp shadows of everyday objects illuminated by visible light. But if one looks carefully enough it will be found that the shadows do not really have sharp boundaries. And in situations where the aperture or obstacle size is comparable to the wavelength, diffraction becomes a major influence in distributing the energy.

The simplest situation mathematically (which we will analyze in detail here) is one where the incoming waves can be approximated by plane waves, and where the intensity pattern is detected far enough from the obstacle or aperture that the waves can again be treated as plane waves. This case is **Fraunhofer diffraction**.

The more complicated case, where the spherical nature of the wavefronts must be taken into account, is **Fresnel diffraction**. We will look at some examples, but not carry out the difficult mathematical analysis for that case.

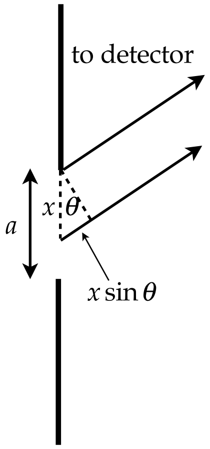
Fraunhofer Diffraction by rectangular slits

To illustrate the phenomenon and the method of analysis, we consider an opaque screen in which has been cut a set of identical parallel long narrow rectangular slits. Light is incident on the screen from the left, at normal incidence. We assume the light to be plane waves, so the rays are parallel. After passing through the slits, the light is diffracted. We seek the intensity of the light moving off in the direction indicated by angle θ.



To calculate the intensity at a distant detector we must find the E-field at that point. This field is the superposition of the fields from all the point sources (according to Huygens's principle) on the part of the wavefront that passes through the slits. These sources emit waves that begin in phase at the slits, but they travel different distances to the detector, so they arrive with different phases and interfere to produce the detected intensity.

First we consider the waves emanating from various point within one of the slits. Let the slits have width *a* and length *L* perpendicular to the page *L* (We assume $L >> a$). Let the distance to the detector from the *top* edge of the *top* slit be *R*. Consider a point within the top slit at distance *x* below the top of that slit. A wave emanating from this point will travel distance $R + x\sin\theta$ to the detector. (See the diagram.)



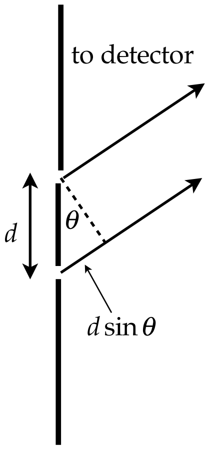
Now consider a strip of area of width dx and length *L*, extending perpendicular to the page and located vertically at the distance *x* from the top of the slit. All the waves in this infinitesimal area travel distance $R + x\sin\theta$ to the detector, so they all arrive in phase with each other. Their total contribution to the E-field at the detector is thus proportional to the area of the strip, and therefore to dx.

We describe this contribution to the detected E-field by

$$dE_1(x) = K\,dx \cdot \cos[k(R + x\sin\theta) - \omega t].$$

(The subscript 1 indicates the top slit.) Here *K* is a constant depending on *L* and the intensity of the incident light. To find the total E-field at the detector due to light from the entire top slit, we integrate this expression over *x* from 0 to the width *a* of the slit. If we had only one slit, this would give the total E-field at the detector.

But we are considering the general case where there are N identical parallel slits separated by distance *d*. Consider light from corresponding points in the top two slits. As the diagram shows, the light from the second slit travels an extra distance $d\sin\theta$ to get to the detector. Thus the wave from this part of the second slit is described by



$$dE_2(x) = K\,dx \cdot \cos[k(R + x\sin\theta + d\sin\theta) - \omega t].$$

For the *n*th slit this contribution is

$$dE_n(x) = K\,dx \cdot \cos[k(R + x\sin\theta + (n-1)d\sin\theta) - \omega t].$$

The total E-field at the detector from all the slits is therefore

$$E(\theta) = K\int_0^a dx \cdot \sum_{n=1}^{N} \cos[k(R + x\sin\theta + (n-1)d\sin\theta) - \omega t]\,.$$

This gives us the field we need. The problem is to evaluate the sum and the integral. It looks formidable, but it turns out not to be too hard.

First we use the Euler trick. We replace the cosines by complex exponentials and use the factoring properties of exponentials:

$$E^c(\theta) = Ke^{i(kR-\omega t)}\int_0^a dx\, e^{ikx\sin\theta} \cdot \sum_{n=1}^{N} e^{ik(n-1)d\sin\theta}\,.$$

Carrying out the sum and the integral are relatively straightforward exercises.

The integral is just an exponential, so it is easy; the sum turns out to be a geometric series, for which there is a simple formula.

We will simply quote the result:

$$E^c(\theta) = Ke^{i(kR-\omega t)}aN\frac{e^{iN\beta}}{e^{i\beta}} \cdot \frac{\sin\alpha}{\alpha} \cdot \frac{\sin N\beta}{N\sin\beta}$$

where we have introduced two angles to shorten the writing:

$$\alpha = \tfrac{1}{2}ka\sin\theta,\ \ \beta = \tfrac{1}{2}kd\sin\theta\,.$$

We are now ready to calculate the intensity. We multiply this expression for $E^c(\theta)$ by its complex conjugate to obtain the squared amplitude of the E-field. The intensity is a constant times this, so we have

$$I(\theta) = \text{const} \cdot \left(\frac{\sin\alpha}{\alpha}\right)^2 \cdot \left(\frac{\sin N\beta}{N\sin\beta}\right)^2\,.$$

The exponentials in the formula for $E^c(\theta)$ disappear when multiplied by their complex conjugates.

To evaluate the constant, we note that as $\theta \to 0$ (which means both $\alpha \to 0$ and $\beta \to 0$) the two factors in the formula both approach 1, so the constant is just $I(0)$. This gives us the final answer:

Fraunhofer pattern for N slits	$I(\theta) = I(0) \cdot \left(\frac{\sin\alpha}{\alpha}\right)^2 \cdot \left(\frac{\sin N\beta}{N\sin\beta}\right)^2$ $\alpha = \tfrac{1}{2}ka\sin\theta,\ \beta = \tfrac{1}{2}kd\sin\theta$

This is a complicated formula, but it contains a great deal of information. The intensity at depends not only on the angle (θ) at which the light is detected, but also on the wavelength of the light (represented by $k = 2\pi / \lambda$), the width of each slit (a), the number (N) of slits, and their separation (d).

The overall amplitude of the waves is proportional to the total area through which the light passes, which in turn is proportional to the slit width a and the number N of slits. This means that $I(0)$ is proportional to $(Na)^2$.

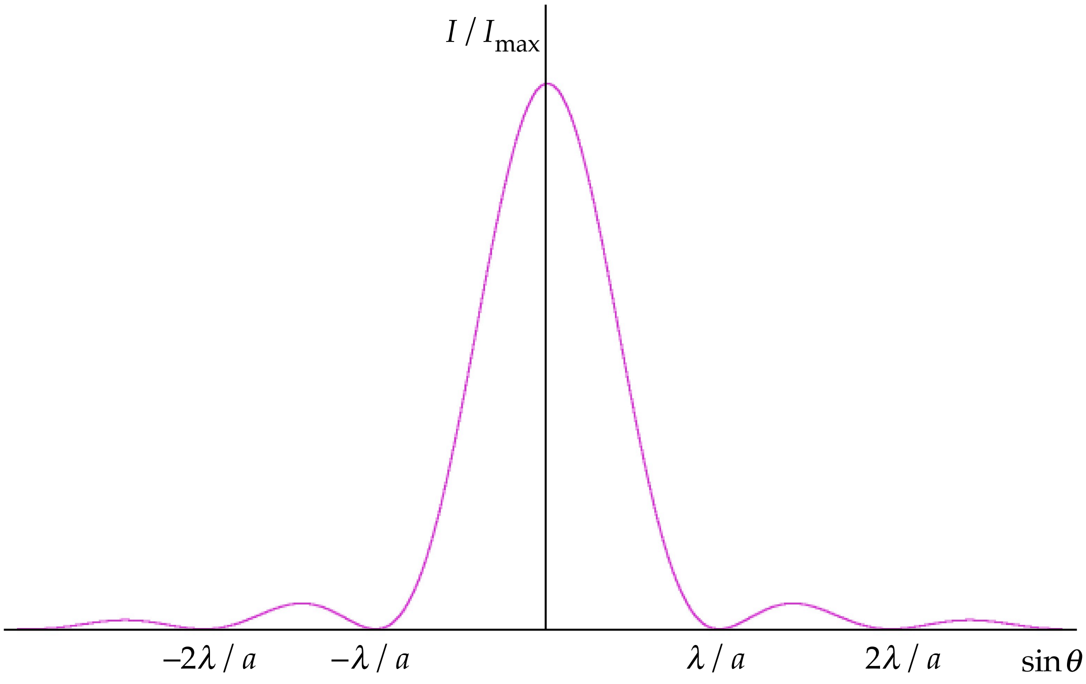
We will examine the pattern described by the equation for various values of N.

Single slit pattern

If $N = 1$ the last factor in the general formula is 1 and we have

Fraunhofer single slit pattern	$I(\theta) = I(0) \cdot \left(\frac{\sin\alpha}{\alpha}\right)^2$

Since $\sin\alpha / \alpha$ has its largest value (1) when $\alpha = 0$ (i.e., when $\theta = 0$), we see that $I(0)$ is the largest intensity for any angle, so we call it $I_{\max}$. The curve shows $I(\theta)$:



The minima in this pattern occur when α is a multiple of π (except zero), i.e., when

$$\text{Minima: } a\sin\theta = \pm\lambda, \pm 2\lambda, \ldots$$

As the slit is made narrower the angular distance between the minima on both sides of the central peak becomes larger: a narrower slit gives a wider pattern, and vice versa.

This behavior is typical of propagation of energy by waves. If light were a stream of classical particles, narrowing the slit would narrow the image on the screen, because all the energy would be concentrated within the boundaries of the geometrical shadow. But because light propagates as a wave, the opposite happens: narrowing the slit produces a greater spread in the image on the screen.

If the aperture is a circular hole instead of a long narrow slit, the formula is more complicated, but the general features are the same. The intensity pattern is a bright central circle peaked at $\theta = 0$, surrounded by a series of concentric alternating dark and bright circular bands of decreasing intensity. The first dark band occurs at an angle given approximately by

$$\text{Circular opening: } \theta_1 \approx 1.22\lambda / D$$

where D is the diameter of the circular opening. This formula is important in determining the resolution of optical instruments with circular apertures.

Multiple slits

If $N > 1$ we must take into account the second factor in the general formula. It gives zero intensity when its numerator vanishes but not its denominator. This occurs if $N\beta$ is a multiple of π but β is not, that is, if

$$N\beta = \pm\pi, \pm 2\pi, \ldots, \pm(N-1)\pi, \pm(N+1)\pi, \ldots$$

These locate the *minima* in the pattern.

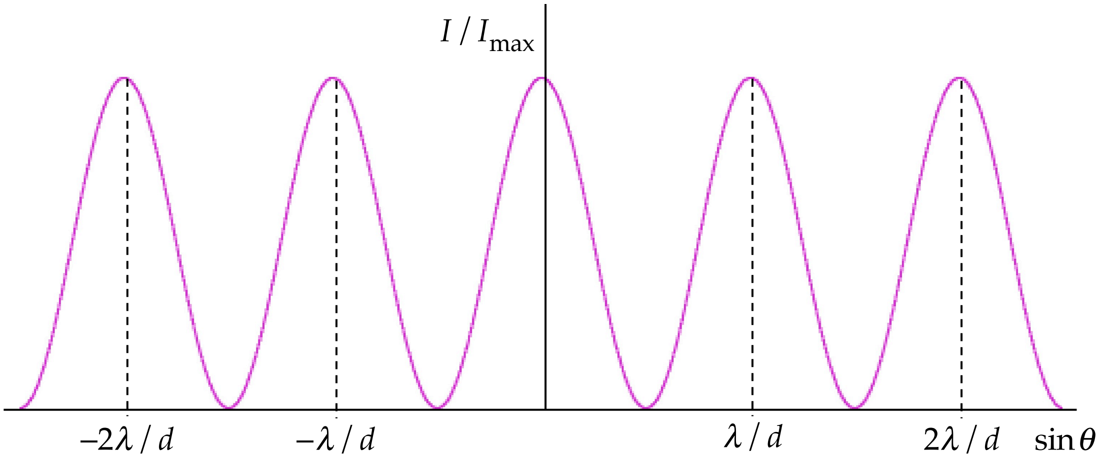
On the other hand, when β is a multiple of π (including zero), the factor becomes 1 and we have a **principal maximum**. The condition $\beta = m\pi$ is usually written out as

Principal maxima	$d\sin\theta = m\lambda, \ m = 0, \pm 1, \pm 2 \ldots.$

The main feature of the pattern consists of the principal maxima, where the peak intensity is the maximum allowed by the $(\sin\alpha / \alpha)^2$ factor. There are $N-1$ minima and $N-2$ smaller "subsidiary" maxima between successive principal maxima.

We will examine the patterns for small values of N, and then look at the case where N is very large.

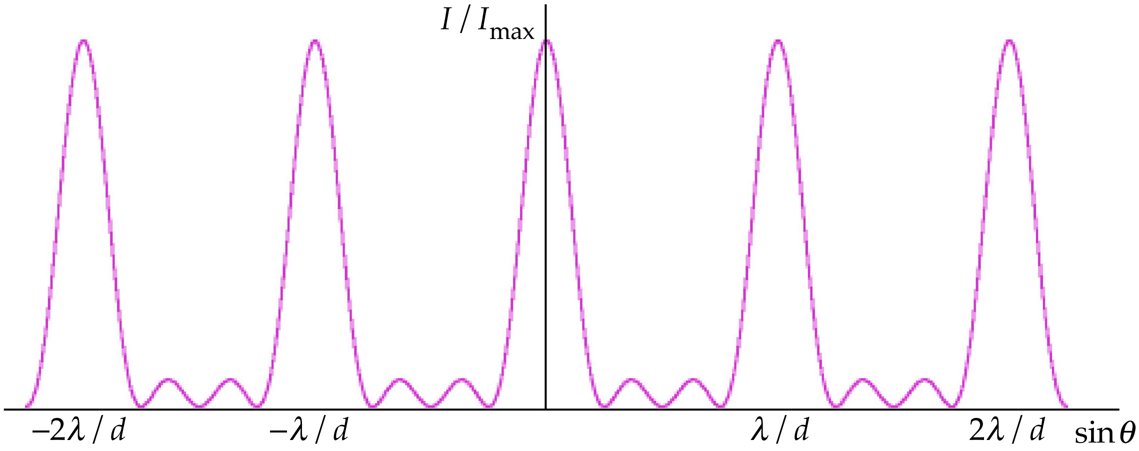
For $N = 2$ there are only principal maxima, with one minimum between each pair. This pattern is shown (we assume a is so small that $\sin\alpha / \alpha \approx 1$).



As the separation d between the slits increases, the peaks move closer together.

If the slit width a is not much smaller than d, then the pattern shown above is "modulated" by the single slit pattern given earlier. The principal maximum peaks to either side of the central one (at $\theta = 0$) are correspondingly reduced in height.

The pattern for $N = 4$ shows the subsidiary maxima and the fact that the principal peaks are narrower. For given d the principal maxima occur at the same angles, regardless of the value of N. (Again we have assumed that $a << d$.)



As the number of slits N is increased, two things happen to the principal maxima:

1. They become narrower. The half-width (the angular distance from a principal maximum peak to the next minimum) is approximately λ / Nd.
2. They become brighter, since $I_{\max}$ is proportional to N^2.

The fraction of the energy in the subsidiary maxima also becomes smaller; for very large N they become so faint that only the principal maxima are visible.

Diffraction gratings

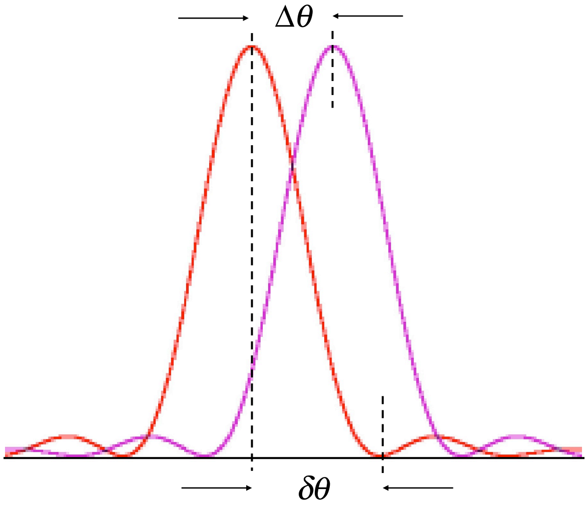
As the number of slits N becomes very large, the principal maxima become very bright and sharp. Knowing d, the location of these maxima can be used to measure the wavelength of the incident light. A diffraction grating is a system with a very large number — of order 10^4 or more — slits (or *rulings*) designed to measure wavelengths.

Since the light emitted by most sources consists of spectral lines (discrete wavelengths), some of which may have nearly the same wavelength, a desirable property of a grating is its ability to distinguish ("resolve") two nearly equal wavelengths. This is called the **resolving power** of the grating, defined as

$$R = \lambda / \Delta\lambda \, .$$

Here λ is the average wavelength in the region and $\Delta\lambda$ is the difference between the closest wavelengths that the grating can clearly distinguish as separate. But under what circumstances can we “clearly” tell that they are separate? We need a criterion.

Consider principal maxima produced by two slightly different wavelengths. Their peaks are located at angles that differ by an *angular separation* $\Delta\theta$. Each peak has a *half-width*, defined as the angular distance from the peak to the first minimum on either side, which we call $\delta\theta$. If $\Delta\theta$ is large enough, the overall intensity pattern clearly reveals that there are two peaks, but if $\Delta\theta$ is too small the peaks may overlap too much to be distinguished. The criterion used to specify how large $\Delta\theta$ must be is due to Rayleigh:

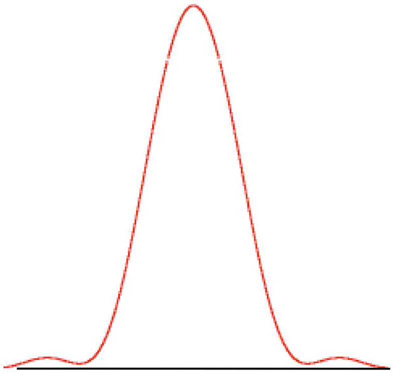


Rayleigh’s criterion	Two diffraction peaks are resolved if the peak separation $\Delta\theta$ is at least equal to the peak half-width $\delta\theta$.

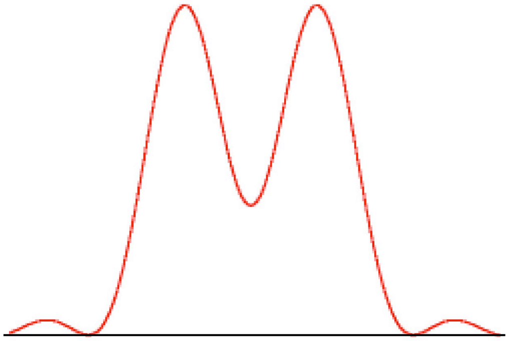
We will apply this criterion to the question of resolving power of a grating, and later to the issue of image resolution with optical instruments.

Shown here are three cases of the total intensity for the two peaks in the drawing above. The curve represents the sum of the intensities of the two peaks, which is what a detector will record.

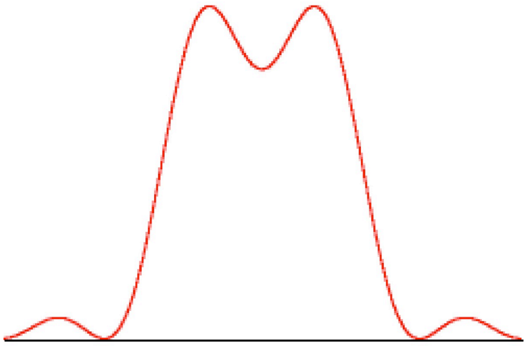
In the top case the peaks are too close together to be resolved. In the middle case they are far apart and are quite well resolved. The bottom case shows the peaks barely resolved.



$\Delta\theta < \delta\theta$



$\Delta\theta > \delta\theta$



$\Delta\theta = \delta\theta$

Now back to diffraction gratings. We consider two wavelengths, λ and $\lambda + \Delta\lambda$. Their *m*th principal maxima will occur at angles θ and $\theta + \Delta\theta$, respectively, where

$$d\sin\theta = m\lambda$$
$$d\sin(\theta + \Delta\theta) = m(\lambda + \Delta\lambda)$$

Here the integer *m* is called the **order number**, and the central maximum at $\theta = 0$ (for all wavelengths) is the case $m = 0$.

If the peaks are barely resolved, then $\theta + \Delta\theta$ must also be the location of the first minimum beyond the *m*th principal maximum for wavelength λ. This minimum occurs when $\beta = (m + 1/N)\pi$, so

$$d\sin(\theta + \Delta\theta) = (m + 1/N)\lambda$$

Comparing with the second equation above, we find a simple formula for the resolving power:

Resolving power of a grating	$R = \lambda / \Delta\lambda = mN$

This shows that to get better resolution it is important to have many rulings, which is why professional gratings typically have $N \sim 10^5$. The spectrum is spread out more in higher orders, which is why *R* depends on *m*; but because $\sin\theta$ cannot exceed 1 there is a limit to how large *m* can be.

Resolution in Optical Instruments

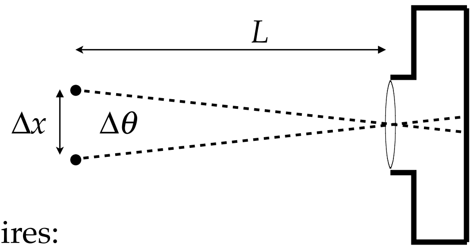
Common optical instruments use circular openings through which light passes. As we have seen, the angular half-width of the Fraunhofer pattern of a circular opening is

$$\delta\theta = 1.22\lambda / D$$

where D is the diameter of the opening.

When such instruments are used to make images, points in the original object become diffraction patterns in the image. If these patterns overlap too much the image does not reveal that the original had separate points: the points are not *resolved* in the image.

In applying Rayleigh's criterion to such instruments, one takes $\Delta\theta$ to be the angle between the rays from the two objects as they enter the instrument through the center of the opening. For the images to be resolved, this angle must be at least as large as $\delta\theta$.



Consider a simple camera, as shown. Two points in the object are separated by distance Δx; the object is at distance L from the cameral lens. The angular separation of the image peaks on the film is the same as the angle subtended at the lens by the object points. For the images to be resolved, Rayleigh's criterion requires:

$$\Delta\theta \geq 1.22\lambda / D.$$

If $\Delta x << L$ we have by the small angle approximations $\Delta\theta \approx \Delta x / L$, so $\Delta x \geq 1.22\lambda L / D$. Rayleigh's criterion specifies a limit on how close two object points can be and still give resolved images.

This restriction applies to microscopes and telescopes, where it can be a serious limitation on the ability of the instrument to provide magnified images.

Fresnel diffraction

As often happens in science, the earliest example of diffraction to be analyzed carefully was not the simplest analytically. Around 1815 Fresnel gave a mathematical theory of diffraction of light waves as they pass by a straight edge obstacle, showing that there are "fringes" (bands of alternating higher and lower intensity) near the edge of the shadow, while some light actually gets into the shadow region where ray geometry would predict it to be dark.

In his analysis, Fresnel could not use the plane wave approximation of Fraunhofer. As a result, the mathematics is much more complicated, but his more general version accounts for phenomena where the source and detector are *not* far from the apertures or obstacles. Because of its mathematical complexity, we will not discuss **Fresnel diffraction** quantitatively.

The mathematician Poisson (who wanted to prove the wave theory wrong) showed that Fresnel's method also predicted a bright spot in the center of the shadow of an opaque disk, and a dark spot in the center of the pattern of light passing through a circular hole. These "absurd" predictions were soon confirmed experimentally by Arago and Fresnel, giving added strength to proponents of the wave theory.

Fresnel was also the first to give the correct formulas for the intensities of the reflected and refracted waves at an interface (long before Maxwell's theory of light as an electromagnetic wave). He also invented a flattened lens, which bears his name, still used in theatrical lighting and lighthouses.